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THE EFFECT OF ELEVATED EXTERNAL PRESSURES ON THE NATURAL CONVECTION OF A LIQUID

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Some technological processes involving the liquid phase proceed at elevated (compared with atmospheric) pressures. These are primarily casting and crystallization under pressure. The structure and properties of cast metals and alloys are determined to a large extent by the regime of crystallization, which (along with temperature and other technological parameters) can be controlled by pressure. In the case of crystallization under pressure, the action of pressure on a crystallizing alloy facilitates the formation of a more homogeneous, fine-grained structure of castings [1].

Heat carriers working at elevated temperatures and pressures are used in some energy plants. Naturally, in this case the heat-carrier characteristics that depend on both temperature and pressure change. To work out a correct design of the operation of the plants, these variations in the thermophysical characteristics of the substance must be taken into account in the basic equations of hydrodynamics and heat transfer.

Under normal conditions, the properties of a liquid are little affected by pressure nonuniformity in a moving medium. If an elevated external pressure is applied to the system, the resulting thermophysical quantities differ markedly from those at atmospheric pressure [2].

As for the temperature dependences, in the majority of papers devoted to free convective flows, it is assumed that the thermophysical characteristics of liquids are constant, except for the density in the buoyancy term in the equation of motion. Various methods are used to take into account the variability of the thermophysical characteristics. The determining-temperature method [3] is most extensively used. This effect, however, can be taken into account more accurately, if temperature variations in the characteristics of liquids are calculated locally in the process of solution.

Whereas the temperature dependences of the thermodynamic characteristics of different substances in the liquid phase are known, these characteristics at elevated pressures have not been adequately studied. The thermodynamic method, using an equation of state and familiar thermodynamic relations, provides information on the characteristics of a substance in the temperature and pressure ranges required for the technological regime considered.

A mathematical simulation of heat- and mass-exchange processes under pressure is performed in the present paper. The processes at atmospheric pressure and an elevated external pressure are compared using a test problem of free laminar convection of a liquid in a square chamber heated from the side. The equations of motion, unlike the conventional equations in the Boussinesq approximation, take into account the temperature dependence of the viscosity coefficient. The thermophysical characteristics entering in the equations are calculated locally in accordance with the varying temperature field. The same characteristics depend parametrically on the applied pressure.

1. Thermodynamic and Thermophysical Characteristics of Water at Atmospheric and Elevated Pressure. Water — the best-investigated substance — was chosen as the working material for the problem. The temperature dependence of water characteristics at atmospheric pressure has been much studied. Figure 1 presents data of the table from [3] divided by the corresponding maximum values $f^0 = f/f_{\max}$. It is seen that β^0 and μ^0 vary substantially (curves 1 and 2), λ^0 varies noticeably, and ρ^0 only slightly (curves

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3 and 4) with increasing temperature t. Figure 1 does not show variation of specific heat c_p^0, because it is too small at the chosen scale.

The thermophysical characteristics of liquids at elevated (as compared with atmospheric) pressure are poorly known. Water is an exception, for which there are tables [4] of quantities $\lambda(T,p)$, $\mu(T,p)$, and $v(T,p)$, where $v = 1/\rho$ is the specific volume. There are data for the thermal compressibility factor $\alpha(T,p)$ [5], but there are no sufficiently reliable data on the functions $c_p(T,p)$ and other quantities.

The missing temperature and pressure dependences of the thermodynamic quantities in the system can be determined by means of an equation of state of the liquid and well-known differential equations of thermodynamics. Since for the liquid phase there is no theoretical equation of state, specific empirical equations [6] are used. The Tait empirical equation of state containing p and T as independent variables is used in the present paper to calculate water characteristics:

$$(p_0 + L)/(p + L) = (v/v_0)^n. \tag{1.1}$$

Here p_0 is the reference (for example, atmospheric) pressure; p is the current elevated pressure, at which the desired quantities are determined; v_0 and v are the specific liquid volumes that correspond to the pressures p_0 and p; n is a constant; and $L(T)$ is an empirical function of temperature, which, as the quantity n, is to be determined from experimental data.

Let us calculate the following thermal (or, more precisely, thermodynamic) coefficients [6]:

isothermal compressibility factor

$$\alpha = -v^{-1}(\partial v/\partial p)_T = [n(p + L)]^{-1}; \tag{1.2}$$

volumetric expansion factor

$$\beta = v^{-1}(\partial v/\partial T)_p = (dL/dt)(p - p_0)/[n(p_0 + L)(p + L)]. \tag{1.3}$$

The variation in the heat capacity at constant pressure $\Delta c_p \equiv c_p(p,T) - c_p(p_0,T)$ with an increase in pressure from p_0 to p can be calculated by integrating the differential thermodynamic relation $(\partial c_p/\partial p)_T = -T(\partial^2 v/\partial T^2)_p$ along the isotherm $T = \text{const}$. As a result, we obtain

$$\Delta c_p = T\frac{d^2L}{dT^2}\frac{n(v_0 - v) - v(p - p_0)/(p_0 + L)}{n - 1} + T\left(\frac{dL}{dT}\right)^2\frac{(p - p_0)^2 v}{n(p_0 + L)^2(p + L)}. \tag{1.4}$$

Determining the constant n and the form of the temperature function $L(T)$ in the Tait equation of state (1.1) and in relations (1.2)–(1.4) is the next step. Zelenetskii et al. [2] processed the data for α of [5] by the least-squares method using expression (1.2), and after appropriate temperature and pressure averaging over the intervals of $0 \leqslant t \leqslant 80$°C and $0.1 \leqslant p \leqslant 100$ MPa, obtained the following values: $n = 6.4$, $L(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$, $a_0 = 3.0504 \cdot 10^8$, $a_1 = 2.1140 \cdot 10^6$, $a_2 = -2.770 \cdot 10^4$, and $a_3 = 87.166$ (L in pascals and t in degrees C).

We compare water-density variations caused by different external effects. When pressure increases from 0.1 to 100 MPa at $t = 20$°C, the water density calculated by Eq. (1.1) increases by 40.4 kg/m^3 (or by 4% of the initial value at atmospheric pressure). When the water is heated at atmospheric pressure from 10 to 100°C, its density decreases by 41.35 kg/m^3, and when ice melts ($t = 0$), the density increases by 83.04 kg/m^3.

Calculations using the Tait equation show that the water density has a weaker dependence on temperature variation as external pressure increases. For example, at $p = 100$ MPa, heating of water from 0 to 80°C decreases its density only by 4.3 kg/m^3, whereas the same heating of water at atmospheric pressure gives 28.05 kg/m^3.

There is a critical temperature T^* at which $dL/dT = 0$. For water, $T^* = 323.1$ K. Since the volumetric expansion factor β is proportional to dL/dT (1.3), the quantity β changes sign with passage through T^*. Thus, at $p > p_0$ and $T > T^*$, the natural convective water flow reverses its direction.

Figure 2 gives the functions $\beta = \beta(p)$ calculated by relation (1.3) for $t = 0$, 10, 20, 40, 60, and 80°C (curves 1–6). A considerable effect of the external pressure and temperature on β and an inversion of β at $t = t^*$ are observed.

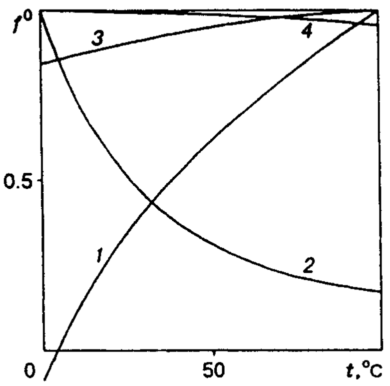

Fig. 1

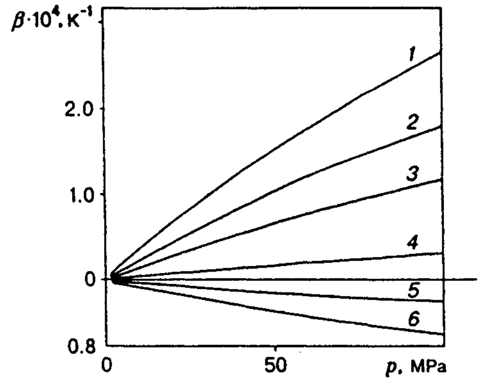

Fig. 2

The heat capacity at constant pressure (1.4) varies only slightly. Thus, at $t = 20$°C and with an increase in pressure by 99 MPa, we have $\Delta c_p = 5.8$ J/(kg · K), which corresponds only to a 0.14% variation.

The thermodynamic method does not make it possible to calculate the transfer coefficients for a liquid at various external pressures and temperatures. The functional relationships $\lambda(p, T)$ and $\mu(p, T)$ can be obtained by approximation of experimental data (for example, by the least-squares method) with the help of polynomials of appropriate degree [2]. In the ranges of $0 \leqslant t \leqslant 90$°C and $0.1 \leqslant p \leqslant 50$ MPa,

$$\lambda/\lambda_0 = [1 + (21.865 - 0.42745t + 7.4163 \cdot 10^{-3}t^2 - 3.5804 \cdot 10^{-5}t^3)10^{-4}p]^{1/2},$$

$$\mu/\mu_0 = 1 + (0.10257t - 3.2653) \cdot 10^{-4}p,$$

where λ_0 and μ_0 are the coefficients at a pressure of 0.1 MPa.

2. Mathematical Model. The thermal-convection equations in the Boussinesq approximation [7] are obtained under the assumption that the thermophysical characteristics of the liquid are constant. Actually, as was shown above, these coefficients depend on temperature and pressure. The viscosity μ varies most substantially with temperature and the heat conductivity λ varies noticeably (see Fig. 1 for relative values).

The convection equations taking into account the temperature dependence of viscosity differ from the conventional Boussinesq equations by the replacement of the viscous force $\mu\Delta v_i$ by $\partial\sigma_{ik}/\partial x_k$ (σ_{ik} is the viscous-stress tensor [7, 8]). Considering the pressure dependence of viscosity parametric and taking into account only $\mu = \mu(T)$, from the Navier–Stokes equation we obtain

$$\begin{gathered}
\rho\Big(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\Big) = -\frac{\partial p}{\partial x} + \mu\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) \\
+2\Big(\frac{d\mu}{dT}\Big)_p\Big[\Big(\frac{\partial T}{\partial x}\Big)\Big(\frac{\partial u}{\partial x}\Big) + \Big(\frac{\partial T}{\partial y}\Big)\Big(\frac{\partial u}{\partial y}\Big)\Big] + \Big(\frac{d\mu}{dT}\Big)_p\Big(\frac{\partial T}{\partial y}\Big)\Big(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\Big), \\
\rho\Big(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\Big) = -\frac{\partial p}{\partial y} + \mu\Big(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\Big) \\
+2\Big(\frac{d\mu}{dT}\Big)_p\Big[\Big(\frac{\partial T}{\partial x}\Big)\Big(\frac{\partial v}{\partial x}\Big) + \Big(\frac{\partial T}{\partial y}\Big)\Big(\frac{\partial v}{\partial y}\Big)\Big] \\
+\Big(\frac{d\mu}{dT}\Big)_p\Big(\frac{\partial T}{\partial x}\Big)\Big(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\Big) - g\rho_0[1 - \beta(T - T_0)].
\end{gathered} \tag{2.1}$$

The heat-transfer equation ignoring dissipation takes the form

$$\rho c_p\Big(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\Big) = \lambda\Big(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\Big) + \Big(\frac{d\lambda}{dT}\Big)_p\Big[\Big(\frac{\partial T}{\partial x}\Big)^2 + \Big(\frac{\partial T}{\partial y}\Big)^2\Big]. \tag{2.2}$$

Here, as for μ in (2.1), the pressure dependence of the heat-conductivity coefficient is parametric, and $\lambda = \lambda(T)$ is explicit. The number of nonlinear terms in Eqs. (2.1) and (2.2) increased in comparison with the standard Boussinesq equations.

The liquid is considered incompressible with respect to hydraulic factors, but it can be compressed, before the onset of the process, by an external elevated pressure which is kept constant during the process.

Let us make the equations dimensionless. Let T_0 be the characteristic temperature. We introduce the following reduced quantities (the ratio of thermophysical quantities at the current temperature T to its values at T_0): $\rho_* = \rho/\rho_0$, $c_{p*} = c_p/c_{p0}$, $\lambda_* = \lambda/\lambda_0$, $\mu_* = \mu/\mu_0$, and $\beta_* = \beta/\beta_0$. To make the equations dimensionless, we choose the following scales: l for the length, $\tau_0 = l^2/\nu_0$ for time, and $\delta T = T_h - T_c$ for the temperature difference ($\nu_0 = \mu_0/\rho_0$ is the characteristic kinematic viscosity of the liquid, and T_h and T_c are the temperatures of the hot and cold walls, respectively).

The stream function

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x}$$

and the vorticity

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\Big(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\Big) \tag{2.3}$$

are introduced in the standard way.

After transformations, we obtain the following dimensionless equation of motion in conservative form

$$\begin{gathered}\frac{\partial\omega}{\partial\tau} + \frac{\partial(u\omega)}{\partial x} + \frac{\partial(v\omega)}{\partial y} = K_1\Big(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2}\Big) + K_2\Big[\Big(\frac{\partial T}{\partial x}\Big)\Big(\frac{\partial\omega}{\partial x}\Big) + \Big(\frac{\partial T}{\partial y}\Big)\Big(\frac{\partial\omega}{\partial y}\Big) \\ +2\Big(\frac{\partial^2 T}{\partial x\partial y}\Big)\Big(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\Big) + \Big(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\Big)\Big(\frac{\partial^2 T}{\partial x^2} - \frac{\partial^2 T}{\partial y^2}\Big)\Big] + K_3\frac{\partial T}{\partial x} \\ [K_1 = \mu_*/\rho_*, \quad K_2 = (d\mu_*/dT)_p/\rho_*, \quad K_3 = \mathrm{Gr}_0\beta_*/\rho_*]\end{gathered} \tag{2.4}$$

and the heat-transfer equation

$$\begin{gathered}\frac{\partial T}{\partial\tau} + \frac{\partial(uT)}{\partial x} + \frac{\partial(vT)}{\partial y} = K_4\Big(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\Big) + K_5\Big[\Big(\frac{\partial T}{\partial x}\Big)^2 + \Big(\frac{\partial T}{\partial y}\Big)^2\Big] \\ [K_4 = \lambda_*/(\rho_* c_{p*}\mathrm{Pr}_0), \quad K_5 = (d\lambda_*/dT)_p/(\rho_* c_{p*}\mathrm{Pr}_0)].\end{gathered} \tag{2.5}$$

Here $\mathrm{Pr}_0 = \nu_0/a_0$ and $\mathrm{Gr}_0 = g\beta_0\delta T l^3/\nu_0^2$ are the Prandtl and Grashof numbers formed of the characteristic thermophysical quantities and constant parameters of the process. Along with them, the dimensionless relative variables K_i ($i = 1,\ldots,5$) that depend locally on temperature enter in the equation. When $K_1 = 1$, $K_2 = 0$, $K_3 = \mathrm{Gr}_0$, $K_4 = 1/\mathrm{Pr}_0$, and $K_5 = 0$, Eqs. (2.4) and (2.5) coincide with the conventional equations of thermal convection.

The problem is formulated as follows. Plane convective motion of the liquid in an infinite cylinder of square section heated from the side is considered. All boundaries of the domain are rigid and fixed. The coordinate origin is placed in the left lower corner. The temperature on the lateral surfaces is constant: $T = 0$ for $x = 0$ and $T = 1$ for $x = 1$, and the horizontal boundaries are adiabatically isolated: $\partial T/\partial y = 0$ for $y = 0$ and 1. The attachment condition is specified at the rigid boundaries.

The problem was solved numerically by the method of grids. Explicit schemes were used to approximate the equations, because "... explicit nonstationary methods are less prone to the instability due to the nonlinearity of the equations, and, hence, they are less sensitive to the initial conditions" [9]. The number of nonlinear terms in the system of equations considered increased in comparison with the Boussinesq equations. The convective terms in the transfer equations were approximated by means of a conservative scheme with donor cells [9]. This scheme is transportable, i.e., a perturbation superimposed on a function is transported due to convection only in the direction of velocity (schemes with central differences do not possess these properties) and partially retains the second order of accuracy which is characteristic of schemes with central differences.

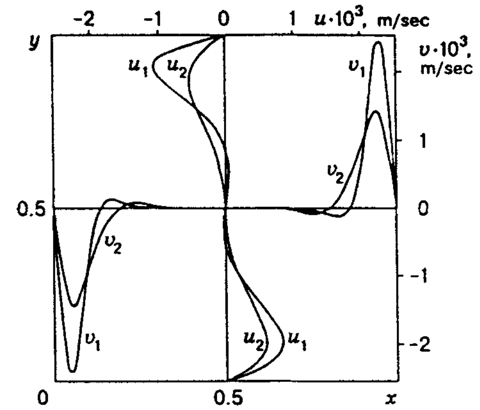

Fig. 3

The Poisson equation (2.3) was solved by the method of successive overrelaxation with optimization of the relaxation-parameter value [10]. The vorticity on the boundary of the domain was calculated by the two-contour method of V. L. Gryaznov and V. I. Polezhayev [11], which takes into account more accurately the attachment condition in the new variable and increases considerably the calculation stability.

3. Calculation Results. The purpose of this investigation is to show the effect of elevated external pressures (the other conditions are fixed) on the liquid-flow pattern and heat exchange. To ignore the inversion of water density at atmospheric pressure near 4°C the temperature interval of the problem was shifted upward of this temperature. In both variants of calculation ($p = 0.1$ and 50 MPa), the temperature was assumed to be $T_c = 283.15$ K on the left boundary and $T_h = 313.15$ K on the right boundary, and the length of the side of the working chamber was $l = 1.8 \cdot 10^{-2}$ m. The Grashof and Prandtl numbers at characteristic temperature $T_0 = 283.15$ K were as follows: $\mathrm{Gr}_0 = 0.77 \cdot 10^5$ and $\mathrm{Pr}_0 = 9.4$ ($p = 0.1$ MPa) and $\mathrm{Gr}_0 = 1.18 \cdot 10^5$ and $\mathrm{Pr}_0 = 9.0$ ($p = 50$ MPa).

We used a 35×35 uniform computation grid. The problem was solved by the pseudotransient method. The vorticity ω (2.4) and temperature T (2.5) calculated at a new time level were compared with their values obtained at the previous level. It was assumed that a steady-state solution was found, if both the relative differences for the vorticity and temperature did not exceed 0.01 for each grid point in the interior computation domain. Iterations in the solution of the Poisson equation (2.3) at each time level were terminated, if the relative variation in the values of the stream function ψ at neighboring iteration levels at each grid point did not exceed 0.001.

Figure 3 presents profiles of the u and v components of the flow velocity for the vertical and horizontal cross sections through the middles of the chamber sides, respectively. Subscript 1 denotes the calculation results for atmospheric pressure, and subscript 2 denotes the results for the elevated external pressure ($p = 50$ MPa).

Figure 4 shows isolines of the relative values of the stream functions $\psi_i^0 = \psi_i/\psi_{\max}$ ($i = 1$ and 2) ($\psi_{\max}$ is the maximum value at $p = 0.1$ MPa) for two variants of the calculation: $i = 1$ and $p = 0.1$ MPa (at the left) and $i = 2$ and $p = 50$ MPa (at the right). The numbering of the curves corresponds to a uniform increase in the values of ψ_i^0. The asterisks indicate the locations of maximum values of the stream functions $(\psi_1^0)_{\max} = 1$ and $(\psi_2^0)_{\max} = 0.767$.

It follows from Figs. 3 and 4 that the flow is of a boundary-layer type. At the vertical isothermal boundaries of the chamber, narrow convective boundary layers form in which the velocity gradients are mainly concentrated. The boundary layers at the horizontal adiabatic planes are less pronounced. The remaining part is the flow core [8]. Flow reversal (Fig. 3) is observed, which is characteristic of flows at large Gr values [3]. This indicates the occurrence of low-rate countercurrents. Convective flow at atmospheric pressure has a more developed boundary-layer regime than fluid flow at $p = 50$ MPa.

The flow rate, apart from $\psi_{\max}$ just considered, can also be estimated by the maximum value of the velocity modulus $|V|_{\max} = \max\{|V_{ij}|\}$ and by the value of the velocity modulus averaged over the

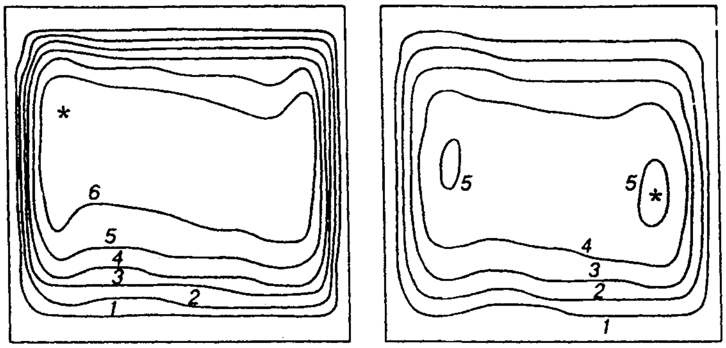

Fig. 4

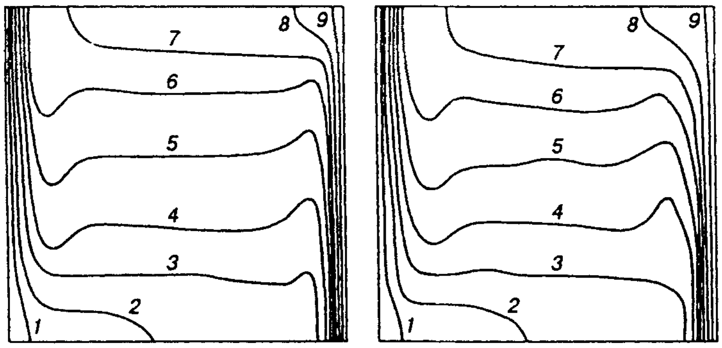

Fig. 5

computation domain $\langle |V| \rangle$. The quantity $|V_{ij}| = \sqrt{(u_{ij}^2 + v_{ij}^2)}$ is the velocity modulus for each point (i, j) of the computation grid. Comparison of the quantities $|V_1|_{\max} = 0.25 \cdot 10^{-2}$ m/sec, $|V_2|_{\max} = 0.15 \cdot 10^{-2}$ m/sec and $\langle |V_1| \rangle = 0.59 \cdot 10^{-3}$ m/sec, and $\langle |V_2| \rangle = 0.42 \cdot 10^{-3}$ m/sec shows that the convective-flow rate of water is higher at atmospheric pressure.

Figure 5 shows isotherms of steady-state flows at $p = 0.1$ MPa (at the left) and $p = 50$ MPa (at the right). The numbering of the curves corresponds to a uniform increase in the values of T. Large temperature gradients near the walls and a constant vertical gradient in the interior domain — so-called stratification — are observed. The flow at atmospheric pressure has a more developed thermal boundary layer. The flow patterns and temperature distributions obtained at $p = 0.1$ MPa are in qualitative agreement with the results of other authors [3, 8].

The dimensionless heat flow through the cold wall of the chamber is equal to 11.8 for $p = 0.1$ MPa and 8.3 for $p = 50$ MPa.

The rate of natural water convection is higher at atmospheric pressure. This is apparently a consequence of the fact mentioned above, i.e., with an increase in the external pressure on the liquid, the dependence of the water density on temperature variation becomes weaker, and this leads to a buoyancy reduction.

In conclusion, we note the following: 1) An approach to the description of heat- and mass-transfer processes under conditions of elevated external pressure is proposed; 2) this approach was applied to a test problem on free laminar convection of a liquid (water) in a square chamber heated from the side; and 3) the difference between the processes at atmospheric and elevated ($p = 50$ MPa) external pressures is shown and estimated quantitatively. In particular, it follows from the results of the work that the rate of thermogravitational convection decreases with an increase in the external pressure (other conditions being the same).

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